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A UNIFIED APPROACH FOR GENERALIZED MINIMUM VARIANCE CONTROLLER FOR LINEAR TIME-VARYING SYSTEMS

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ABSTRACT

We study single input and single output stochastic linear time-varying systems and develop a unified approach for generalized minimum variance controllers. The plant to be controlled is a time-varying controlled autoregressive moving average model and the performance index includes both tracking error variance of filtered plant output and a quadratic function of filtered plant input. Both the input and output filters are linear time-varying.

KEYWORDS: Adaptive Control, Generalized Minimum Variance Control, Stochastic Control & Time-Varying Systems

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INTRODUCTION

Generalized minimum variance controllers (GMVCs) are flexible controllers for optimal control of stochastic plants. They have seen many applications [1-3]. Various GMVCs have been developed based on different control strategies in order to meet various requirements of industrial applications [4-8]. The performance index of a GMVC is a cost functional that is a tracking error variance of filtered plant output penalized by a quadratic function of filtered plant input. The basic difference in different approaches is in the choice of input/output filters and weightings for the tracking error variance and the quadratic input function. The input and output filters can be used to modify the closed-loop dynamics for desired transient performance and the weightings can be used for specification of compromise between the output tracking accuracy and magnitude of fluctuation in plant input.

In this paper, we develop a unified approach to various GMVCs for both linear time invariant (LTI) and linear time varying (LTV) single input and single output (SISO) systems by introducing general form of LTV filters for both plant input and output in the cost functional such that various GMVCs are special cases of this LTV GMVC. We will also provide closed-loop dynamics and stability analysis for the unified approach of GMVC. The plants to be controlled is described using an LTV controlled autoregressive moving average (CARMA) model and the filters are described using LTV transfer operators.

The reminder of this paper is organized as the following. Section 2 describes the LTV SISO plants and the GMVC performance index. Section 3 develops the unified LTV GMVC and Section 4 presents a simulation example. Finally, Section 5 concludes this paper.

LTV PLANT AND CONTROL OBJECTIVE

The LTV systems are described using the following SISO CARMA model.

$$y(k+d) = A^{-1}(k,q^{-1})[B(k,q^{-1})u(k) + C(k,q^{-1})w(k+d)] \tag{1}$$

where *u(k)* and *y(k)* are input and output, *d* is a positive integer representing the input/output delay, *w(k)* is an independent Gaussian noise with zero mean and a uniformly bounded time-varying variance, q^{-1} is the one-step-delay operator satisfying $q^{-1}f(k)g(k)=f(k-1)q^{-1}g(k)=f(k-1)g(k-1)$. In the CARMA model $A(k,q^{-1})$, $B(k,q^{-1})$ and $C(k,q^{-1})$ are LTV moving average operators (MAO's) having the following form.

$$K(k,q^{-1}) = k_0(k) + k_1(k)q^{-1} + ... + k_{nk}(k)q^{-nk} \tag{2}$$

where $K(k,q^{-1}) = A(k,q^{-1}),\ B(k,q^{-1})$ and $C(k,q^{-1})$ are polynomials in the delay operator q^{-1} and $a_0(k) = c_0(k) = 1$. The LTV MAO's are natural extension of the moving average operators from the LTI transfer functions for LTV CARMA models, where the constant coefficients in the LTI polynomials are replaced using time-varying parameters. It is assumed that all the time-varying parameters of the LTV polynomials are uniformly bounded, i.e. $|k_i(k)| < c < 0$ for a constant *c* and all the *i* and *k.* It is also assumed that the time delay between the plant input and output is time-invariant, i.e. $|b_0(k)| > c > 0,$ for a constant *c* and all *k.* The inversion operation of the MAO is an autoregressive operator (ARO) denoted by $A^{-1}(k,q^{-1})$. A LTV ARO is also a natural extension of an LTI ARO of LTI transfer functions. Detailed properties of the LTV ARO can be found in [9]. For the design of the LTV GMVC it is assumed that both $A^{-1}(k,q^{-1})$ and $C^{-1}(k,q^{-1})$ are exponentially stable.

Given a uniformly bounded reference *z(k)* the generalised minimum variance control objective is to find an LTV controller that generates a sequence of control variable *u(k)* such that the following generalised minimum variance cost functional is minimized.

$$\begin{aligned} J(k+d) = E(&|P(k,q^{-1})y(k+d) - Q(k,q^{-1})z(k)|^2 \\ &+ V(k)|R(k,q^{-1})u(k)|^2 / D(k)) \end{aligned} \tag{3}$$

where *D(k)={y(k), u(k), y(k-1), u(k-1), …}* is the set of input and output data up to and including the current time *k* and *E(X/D(k))* is the operator for mathematical expectation of *X* conditioned on *D(k), V(k)>c>0,* for a constant *c* and all *k,* is for the weighting between the tracking error and the magnitude of input. The LTV MAO's $P(k,q^{-1}),\ Q(k,q^{-1})$ and $R(k,q^{-1})$ are the LTV filters for the output, reference and input. They have the following forms.

$$K(k,q^{-1}) = K_n(k,q^{-1})K_d^{-1}(k,q^{-1}) \tag{4}$$

where

$$K(k,q^{-1}) = P(k,q^{-1}),\ Q(k,q^{-1}),\ R(k,q^{-1}) \tag{5}$$

and

$$K_n(k,q^{-1}) = k_{n0}(k) + k_{n1}(k)q^{-1} + k_{n2}(k)q^{-2} + \ldots + k_{np}(k)q^{-np} \tag{6}$$

$$K_d(k,q^{-1}) = 1 + k_{d1}(k)q^{-1} + k_{d2}(k)q^{-2} + \ldots + k_{dq}(k)q^{-dq}. \tag{7}$$

It is assumed that $k_{n0}(k)$ is uniformly bounded away from zero and all the parameters in (6) and (7) are uniformly bounded away from infinite. The three filters are LTV moving average and autoregressive operators that are natural extensions of the LTI filters from LTI GMVCs for LTV GMVCs.

Generalized Minimum Variance Controller

We first develop a minimum variance predictor (MVP) for the filtered output

$$\psi(k+d) = P(k,q^{-1})y(k+d). \tag{8}$$

The cost functional is

$$J_p(k+d) = E(|\, P(k,q^{-1})y(k+d) - \hat{\psi}(k+d)\,|^2 \,/\, D(k)) \tag{9}$$

Where $\hat{\psi}(k+d)$ is the d-step-ahead prediction of $\psi(k)$. Left multiplying (1) using $P(k,q^{-1})$ and noting (8) we have

$$\begin{aligned}\psi(k+d) &= P(k,q^{-1})A^{-1}(k,q^{-1})B(k,q^{-1})u(k) \\ &\quad + P_n(k,q^{-1})P_d^{-1}(k,q^{-1})A^{-1}(k,q^{-1})C(k,q^{-1})w(k+d).\end{aligned} \tag{10}$$

The filtered output has two components. The first is the response to the input and the second is caused by the noise. The last term in the above equation is the component caused by the noise.

Noting the last term in (10) and applying long division we have

$$\begin{aligned}&P_d^{-1}(k,q^{-1})A^{-1}(k,q^{-1})C(k,q^{-1})w(k+d) \\ &= [A(k,q^{-1})P_d(k,q^{-1})]^{-1}C(k,q^{-1})w(k+d) \\ &= \{F(k,q^{-1}) + [A(k,q^{-1})P_d(k,q^{-1})]^{-1}G(k,q^{-1})q^{-d}\}w(k+d) \\ &= F(k,q^{-1})w(k+d) + [A(k,q^{-1})P_d(k,q^{-1})]^{-1}G(k,q^{-1})w(k)\end{aligned} \tag{11}$$

Where

$$F(k,q^{-1}) = 1 + f_1(k)q^{-1} + f_2(k)q^{-2} + \ldots + f_{d-1}(k)q^{-d+1} \tag{12}$$

Is the quotient of the following long division

$$\begin{aligned}&[A(k,q^{-1})P_d(k,q^{-1})]^{-1}C(k,q^{-1}) \\ &= F(k,q^{-1}) + [A(k,q^{-1})P_d(k,q^{-1})]^{-1}G(k,q^{-1})q^{-d}\end{aligned} \tag{13}$$

Where

$$G(k,q^{-1}) = g_0(k) + g_1(k)q^{-1} + ... + g_s(k)q^{-s} \tag{14}$$

Is the remainder. The noise component can be further divided into two parts. The first is caused by the noise to occur in the future and the second is caused by the current and past noise. The future part is completely unknown and the current and past part can be estimated using the data set *D(k).* Left multiplying $P_n(k,q^{-1})$ on both sides of equation (11) we have

$$\begin{aligned} &P_n(k,q^{-1})P_d^{-1}(k,q^{-1})A^{-1}(k,q^{-1})C(k,q^{-1})w(k+d) \\ &\quad = P_n(k,q^{-1})F(k,q^{-1})w(k+d) \\ &\quad + P_n(k,q^{-1})[A(k,q^{-1})P_d(k,q^{-1})]^{-1}G(k,q^{-1})w(k). \end{aligned} \tag{15}$$

Letting

$$P_n(k,q^{-1})F(k,q^{-1}) = H(k,q^{-1}) + L(k,q^{-1})q^{-d} \tag{16}$$

Where

$$H(k,q^{-1}) = 1 + h_1(k)q^{-1} + ... + h_{d-1}(k)q^{-d+1} \tag{17}$$

Has all the terms that have delays less than *d* in $P_n(k,q^{-1})F(k,q^{-1})$ and $L(k,q^{-1})q^{-d}$ has all the rest terms, which have delays greater than *d-1.* Substituting (16) into (15) and noting (10) we have

$$\begin{aligned} &\psi(k+d) = P(k,q^{-1})A^{-1}(k,q^{-1})B(k,q^{-1})u(k) \\ &\quad + H(k,q^{-1})w(k+d) + [L(k,q^{-1}) \\ &\quad + P_n(k,q^{-1})P_d^{-1}(k,q^{-1})A^{-1}(k,q^{-1})G(k,q^{-1})]w(k) \end{aligned} \tag{18}$$

Where $H(k,q^{-1})w(k+d)$ is the response due to future noise and $[L(k,q^{-1}) + P(k,q^{-1})A^{-1}(k,q^{-1})G(k,q^{-1})]w(k)$ has all the past and current noises for the filtered output. Taking mathematical expectation conditioned on *D(k)* on both sides of the above equation we have the following *d*-step-ahead minimum variance prediction of the filtered plant output.

$$\begin{aligned} &\hat{\psi}(k+d) = P(k,q^{-1})A^{-1}(k,q^{-1})B(k,q^{-1})u(k) \\ &\quad + [L(k,q^{-1}) + P(k,q^{-1})A^{-1}(k,q^{-1})G(k,q^{-1})]w(k) \\ &\quad = P(k,q^{-1})[A^{-1}(k,q^{-1})B(k,q^{-1})u(k) \\ &\quad + A^{-1}(k,q^{-1})G(k,q^{-1})w(k)] + L(k,q^{-1})w(k) \\ &\quad = P(k,q^{-1})\hat{y}(k+d) + L(k,q^{-1})w(k) \end{aligned} \tag{19}$$

Where

$$\begin{aligned} &\hat{y}(k+d) = A^{-1}(k,q^{-1})B(k,q^{-1})u(k) \\ &\quad + A^{-1}(k,q^{-1})G(k,q^{-1})w(k) \end{aligned} \tag{20}$$

Is the *d*-step-ahead minimum variance prediction of the plant output *y(k)* [10]. The above equation shows that the minimum variance prediction of the filtered output is not the filtered minimum variance prediction of the plant output. The difference between the two is $L(k,q^{-1})w(k)$.

GMVC Theorem

If the LTV ARO's $A^{-1}(k,q^{-1})$, $C^{-1}(k,q^{-1})$ and $P^{-1}(k,q^{-1})$ are exponentially stable, the LTV GMVC for the CARMA model (1) is given by

$$\begin{aligned}\hat{w}(k) &= C^{-1}(k-d,q^{-1})A(k-d,q^{-1})y(k)\\ &\quad -C^{-1}(k-d,q^{-1})B(k-d,q^{-1})u(k)\end{aligned} \tag{21}$$

$$\begin{aligned}&[B(k,q^{-1})+A(k,q^{-1})P^{-1}(k,q^{-1})\frac{r_{n0}(k)V(k)}{p_{n0}(k)b_0(k)}R(k,q^{-1})]u(k)\\ &\quad = A(k,q^{-1})P^{-1}(k,q^{-1})Q(k,q^{-1})z(k)\\ &\quad -[A(k,q^{-1})P^{-1}(k,q^{-1})L(k,q^{-1})+G(k,q^{-1})]\hat{w}(k).\end{aligned} \tag{22}$$

Proof. Subtracting (18) from (19) we have

$$\hat{\psi}(k+d)=\psi(k+d)-H(k,q^{-1})W(k+d). \tag{23}$$

Substituting (23) into the performance index (3) we have

$$\begin{aligned}J(k+d) &= |\hat{\psi}(k+d)-Q(k,q^{-1})z(k)|^2\\ &\quad +V(k)\,|R(k,q^{-1})u(k)|^2+E(|H(k,q^{-1})w(k+d)|^2).\end{aligned} \tag{24}$$

It follows that

$$\begin{aligned}\frac{\partial J(k+d)}{\partial u(k)} &= 2p_{n0}(k)b_0(k)[\hat{\psi}(k+d)\\ &\quad -Q(k,q^{-1})z(k)]+2r_{n0}(k)V(k)R(k,q^{-1})u(k)\end{aligned} \tag{25}$$

and

$$\frac{\partial^2 J(k+d)}{\partial U^2(k)}=2p_{n0}^2(k)b_0^2(k)+2V(k)r_{n0}^2(k)>0. \tag{26}$$

Noting the above equation, we know that the optimal control *u(k)* exists and can be determined by equating (25) to zero. Substituting (19) into (25) and let it equate zero we have

$$\begin{aligned}&P(k,q^{-1})A^{-1}(k,q^{-1})B(k,q^{-1})u(k)+[L(k,q^{-1})\\ &\quad +P(k,q^{-1})A^{-1}(k,q^{-1})G(k,q^{-1})]w(k)\\ &\quad -Q(k,q^{-1})z(k)+\frac{r_{n0}(k)V(k)}{p_{n0}(k)b_0(k)}R(k,q^{-1})u(k)=0.\end{aligned} \tag{27}$$

Solving for *u(k)* we have

$$\begin{aligned}&[P(k,q^{-1})A^{-1}(k,q^{-1})B(k,q^{-1})+\frac{r_{n0}(k)V(k)}{p_{n0}(k)b_0(k)}R(k,q^{-1})]u(k)\\ &\quad = Q(k,q^{-1})z(k)-[L(k,q^{-1})\\ &\quad +P(k,q^{-1})A^{-1}(k,q^{-1})G(k,q^{-1})]w(k).\end{aligned} \tag{28}$$

Left dividing (28) using $P(k,q^{-1})A^{-1}(k,q^{-1})$ we have the optimal control law as follows.

$$\begin{aligned}&[B(k,q^{-1})+A(k,q^{-1})P^{-1}(k,q^{-1})\frac{r_{n0}(k)V(k)}{p_{n0}(k)b_0(k)}R(k,q^{-1})]u(k))\\&=A(k,q^{-1})P^{-1}(k,q^{-1})Q(k,q^{-1})z(k)\\&-[A(k,q^{-1})P^{-1}(k,q^{-1})L(k,q^{-1})+G(k,q^{-1})]w(k)\end{aligned}\tag{29}$$

Where the noise *w(k)* can be estimated based on (1) using (21). Replacing *w(k)* using its estimate we have the GMVC controller (22). Comparing (1) with (21) we have

$$C(k,q^{-1})\tilde{w}(k+d)=0\tag{30}$$

Where

$$\tilde{w}(k+d)=w(k+d)-\hat{w}(k+d)\tag{31}$$

Is the estimation error that will always decay exponentially to zero regardless initial conditions because of exponential stability of $C^{-1}(k,q^{-1})$. From (22) and (31) we have

$$\begin{aligned}&[B(k,q^{-1})+A(k,q^{-1})P^{-1}(k,q^{-1})\frac{r_{n0}(k)V(k)}{p_{n0}(k)b_0(k)}R(k,q^{-1})]u(k))\\&=A(k,q^{-1})P^{-1}(k,q^{-1})Q(k,q^{-1})z(k)\\&-[A(k,q^{-1})P^{-1}(k,q^{-1})L(k,q^{-1})\\&+G(k,q^{-1})][\hat{w}(k)-w(k)+w(k)]\\&=A(k,q^{-1})P^{-1}(k,q^{-1})Q(k,q^{-1})z(k)\\&+[A(k,q^{-1})P^{-1}(k,q^{-1})L(k,q^{-1})+G(k,q^{-1})][\tilde{w}(k)-w(k)].\end{aligned}\tag{32}$$

Noting (1), (31) and (32) we have the following closed-loop equation for the LTV GMVC.

$$\begin{aligned}&\begin{bmatrix}C(k-d,q^{-1}) & 0 & 0\\ -E(k,q^{-1}) & T(k,q^{-1}) & 0\\ 0 & -B(k-d,q^{-1})q^{-d} & A(k-d,q^{-1})\end{bmatrix}\begin{bmatrix}\tilde{w}(k)\\ u(k)\\ y(k)\end{bmatrix}\\&=\begin{bmatrix}0 & 0\\ -E(k,q^{-1}) & A(k,q^{-1})P^{-1}(k,q^{-1})Q(k,q^{-1})\\ C(k-d,q^{-1}) & 0\end{bmatrix}\begin{bmatrix}w(k)\\ z(k)\end{bmatrix}.\end{aligned}\tag{33}$$

Where

$$\begin{aligned}&T(k,q^{-1})=B(k,q^{-1})\\&\quad+A(k,q^{-1})P^{-1}(k,q^{-1})\frac{r_{n0}(k)V(k)}{p_{n0}(k)b_0(k)}R(k,q^{-1})\\&E(k,q^{-1})=A(k,q^{-1})P^{-1}(k,q^{-1})L(k,q^{-1})+G(k,q^{-1}).\end{aligned}\tag{34}$$

The closed-loop equation of the LTV GMVC shows that the closed-loop stability is determined by the following LTV matrix ARO.

$$M^{-1}(k,q^{-1}) = \begin{bmatrix} C(k-d,q^{-1}) & 0 & 0 \\ -E(k,q^{-1}) & T(k,q^{-1}) & 0 \\ 0 & -B(k-d,q^{-1})q^{-d} & A(k-d,q^{-1}) \end{bmatrix}^{-1}. \tag{35}$$

Noting that $M(k,q^{-1})$ is lower triangular and both $A^{-1}(k,q^{-1})$ and $C^{-1}(k,q^{-1})$ are exponentially stable we know that the closed-loop system is exponentially stable if and only if $T^{-1}(k,q^{-1})$ is exponentially stable. Equation (34) shows that exponential stability of $T^{-1}(k,q^{-1})$ can be modified by choosing the weighting *V(k)* and the LTV filters $P(k,q^{-1})$ and $R(k,q^{-1})$.

SIMULATION

We consider the first order SISO LTV CARMA model

$$y(k+2)+a(k)y(k+1)=u(k)+b(k)u(k-1)+w(k+2)+c(k)w(k+1) \tag{36}$$

Where *w(k)* is an independent, stationary and Gaussian white noise with zero mean and unit variance. The plant parameters are

$$\begin{aligned} a(k) &= \begin{cases} 0.4(1-0.8e^{-k}) & 20i < k \le 2(i+1) \\ -0.4(1-0.8e^{-k}) & 20(i-1) < k \le 20i \end{cases} \\ b(k) &= 1.3(1+\sin(0.4\pi k)) \\ c(k) &= \begin{cases} 0.6\dfrac{k}{k+1} & 10i < k \le 10(i+1) \\ -0.6\dfrac{k}{k+1} & 10(i-1) < k \le 10i \end{cases} \end{aligned} \tag{37}$$

Where *i=0, 1, 2,.....* We choose *V(k)=1* and

$$\begin{aligned} P(k,q^{-1}) &= Q(k,q^{-1}) = (1+0.2q^{-1})(1+0.1q^{-1})^{-1} \\ R(k,q^{-1}) &= (1-0.3q^{-1})(1+0.5q^{-1})^{-1}. \end{aligned} \tag{38}$$

Figure 1 shows the time-varying plant parameters. Figure 2 illustrates the plant output and the reference showing the LTV GMVC is able to stabilize the system and drive the output to follow the reference. Figure 3 shows the control variable *u(k).*

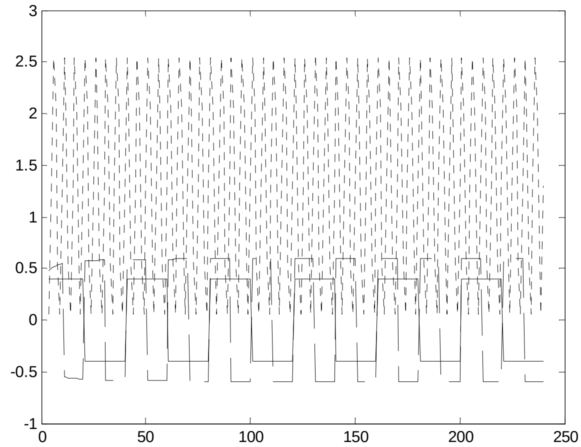

Figure 1: Time-Varying Plant Parameters *a(k)* __ __, *b(k)* ------, and *c(k)*______.

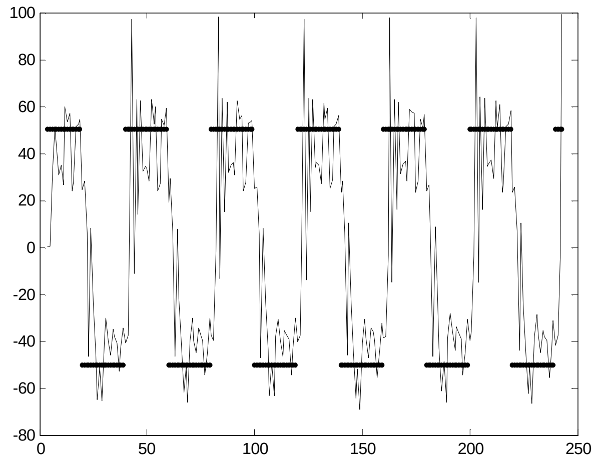

Figure 2: Plant Output and Reference ____

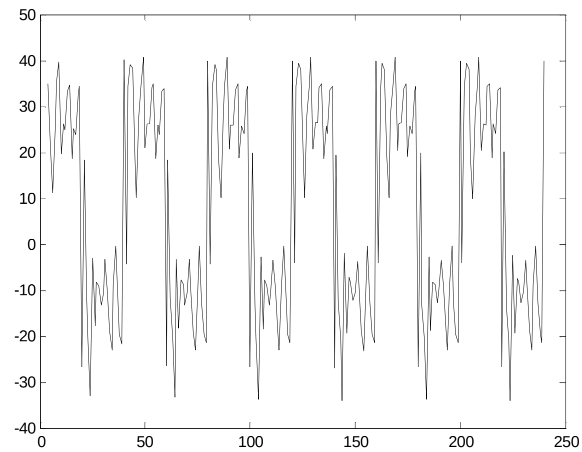

Figure 3: Plant Input *u(k)*

CONCLUSIONS

An LTV GMVC has been developed for SISO LTV stochastic systems. It employs LTV filters in the form of moving average and autoregressive operators for the plant output and input providing a unified approach of LTV GMVCs for SISO CARMA models. It can be applied to a wide class of LTV systems.

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